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Quadratic Concepts -- Multiplying Binomials

This module teaches about multiplying binomials. Specifically about common patterns that can be memorized and using the "FOIL" method.

The following three formulae should be memorized.

Equation:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Equation:

$$(x - a)^2 = x^2 - 2ax + a^2$$

Equation:

$$(x + a)(x - a) = x^2 - a^2$$

It is important to have these three formulae on the top of your head. It is also nice to be able to show **why** these formulae work, for instance by using FOIL. But the most important thing of all is knowing **what these three formulae mean, and how to use them.**

These three are all “algebraic generalizations,” as discussed in the first unit on functions. That is, they are equations that hold true for **any** values of x and a . It may help if you think of the second equation above as standing for:

$(\text{Anything} - \text{Anything Else})^2 = \text{Anything}^2 - 2(\text{Anything Else})^2$ For instance, suppose the **Anything** (or x) is 5, and the **Anything Else** (or a) is 3.

Example:

$(x - a)^2 = x^2 - 2ax + a^2$, **when** $x = 5$, $a = 3$.

- $5 - 3^2 \stackrel{?}{=} 5^2 - 2(3)(5) + 3^2$
- $2^2 \stackrel{?}{=} 25 - 30 + 9$
- $4 = 4$

It worked! Now, let's leave the **Anything** as x , but play with different values of a .

Example:

More examples of $(x - a)^2 = x^2 - 2ax + a^2$

Equation:

$$\begin{aligned}
 a = 1 : \quad (x - 1)^2 &= x^2 - 2x + 1 \\
 a = 2 : \quad (x - 2)^2 &= x^2 - 4x + 4 \\
 a = 3 : \quad (x - 3)^2 &= x^2 - 6x + 9 \\
 a = 5 : \quad (x - 5)^2 &= x^2 - 10x + 25 \\
 a = 10 : \quad (x - 10)^2 &= x^2 - 20x + 100
 \end{aligned}$$

Once you've seen a few of these, the pattern becomes evident: the number **doubles** to create the middle term (the coefficient of x), and **squares** to create the final term (the number).

The hardest thing about this formula is **remembering to use it**. For instance, suppose you are asked to expand:

Equation:

$$(2y - 6)^2$$

There are three ways you can approach this.

$(2y - 6)^2$, computed three different ways		
Square each term	FOIL	Using the formula above
$ \begin{aligned} &(2y - 6)^2 \\ &(2y)^2 - 2(6)(2y) + 6^2 \\ &4y^2 - 24y + 36 \end{aligned} $	$ \begin{aligned} &(2y - 6)(2y - 6) \\ &(2y)(2y) - (2y)6 - (2y)6 + 36 \\ &4y^2 - 12y - 12y + 36 \\ &4y^2 - 24y + 36 \end{aligned} $	$ \begin{aligned} &(2y - 6)^2 \\ &(2y)^2 - 2(6)(2y) + 6^2 \\ &4y^2 - 24y + 36 \end{aligned} $

Did it work? If a formula is true, it should work for any y -value; let's test each one with $y = 5$. (Note that the second two methods got the same answer, so we only need to test that once.)

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$(2y - 6)^2 \stackrel{?}{=} 4y^2 - 36$	$(2y - 6)^2 \stackrel{?}{=} 4y^2 - 24y + 36$
$(2 \cdot 5 - 6)^2 \stackrel{?}{=} 4y^2 - 36$	$(2 \cdot 5 - 6)^2 \stackrel{?}{=} 4(5)^2 - 24 \cdot 5 + 36$
$(10 - 6)^2 \stackrel{?}{=} 100 - 36$	$(10 - 6)^2 \stackrel{?}{=} 100 - 120 + 36$
$4^2 \stackrel{?}{=} 64 \times$	$4^2 \stackrel{?}{=} 16 \checkmark$

We conclude that squaring each term individually does not work. The other two methods both give the same answer, which works.

The first method is the easiest, of course. And it **looks** good. $(2y)^2$ is indeed $4y^2$. And 6^2 is indeed 36. But as you can see, it led us to a **false answer**—an algebraic generalization that did not hold up.

I just can't stress this point enough. It sounds like a detail, but it causes errors all through Algebra II and beyond. **When you're adding or subtracting things, and then squaring them, you can't just square them one at a time.** Mathematically, $(x + a)^2 \neq x^2 + a^2$. You can confirm this with numbers all day. $(7 + 3)^2 = 100$, but $7^2 + 3^2 = 58$. They're not the same.

So that leaves the other two methods. FOIL will never lead you astray. But the third approach, the formula, has three distinct advantages.

1. The formula is faster than FOIL.
2. Using these formulae is a specific case of the vital mathematical skill of using **any** formula—learning how to plug numbers and variables into some equation that you've been given, and therefore understanding the abstraction that formulae represent.
3. Before this unit is done, we will be completing the square, which requires running that particular formula **backward**—which you cannot do with FOIL.

Quadratic Concepts -- Factoring

This module discusses how to solve quadratic equations by factoring.

When we multiply, we put things together: when we factor, we pull things apart. Factoring is a critical skill in simplifying functions and solving equations.

There are four basic types of factoring. In each case, I will start by showing a multiplication problem—then I will show how to use factoring to **reverse** the results of that multiplication.

“Pulling Out” Common Factors

This type of factoring is based on the **distributive property**, which (as you know) tells us that:

Equation:

$$2x(4x^2 - 7x + 3) = 8x^3 - 14x^2 + 6x$$

When we factor, we do that in reverse. So we would start with an expression such as $8x^3 - 14x^2 + 6x$ and say “Hey, **every one** of those terms is divisible by 2. Also, **every one** of those terms is divisible by x . So we “factor out,” or “pull out,” a $2x$.

Equation:

$$8x^3 - 14x^2 + 6x = 2x(__ - __ + __)$$

For each term, we see what happens when we **divide** that term by $2x$. For instance, if we divide $8x^3$ by $2x$ the answer is $4x^2$. Doing this process for each term, we end up with:

Equation:

$$8x^3 - 14x^2 + 6x = 2x(4x^2 - 7x + 3)$$

As you can see, this is just what we started with, but in reverse. However, for many types of problems, this factored form is easier to work with.

As another example, consider $6x + 3$. The common factor in this case is 3. When we factor a 3 out of the $6x$, we are left with $2x$. When we factor a 3 out of the 3, we are left with...what? Nothing? No, we are left with 1, since we are **dividing** by 3.

Equation:

$$6x + 3 = 3(2x + 1)$$

There are two key points to take away about this kind of factoring.

1. This is the simplest kind of factoring. Whenever you are trying to factor a complicated expression, **always begin** by looking for **common factors** that you can pull out.
2. A common factor must be common to **all** the terms. For instance, $8x^3 - 14x^2 + 6x + 7$ has no common factor, since the last term is not divisible by either 2 or x .

Factoring Perfect Squares

The second type of factoring is based on the “squaring” formulae that we started with:

Equation:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Equation:

$$(x - a)^2 = x^2 - 2ax + a^2$$

For instance, if we see $x^2 + 6x + 9$, we may recognize the signature of the first formula: the middle term is three **doubled**, and the last term is three

squared. So this is $(x + 3)^2$. Once you get used to looking for this pattern, it is easy to spot.

Equation:

$$x^2 + 10x + 25 = (x + 5)^2$$

Equation:

$$x^2 + 2x + 1 = (x + 1)^2$$

And so on. If the middle term is **negative**, then we have the second formula:

Equation:

$$x^2 - 8x + 16 = (x - 4)^2$$

Equation:

$$x^2 - 14x + 49 = (x - 7)^2$$

This type of factoring only works if you have **exactly this case**: the middle number is something **doubled**, and the last number is that same something **squared**. Furthermore, although the **middle** term can be either positive or negative (as we have seen), the **last** term cannot be negative.

All this may make it seem like such a special case that it is not even worth bothering about. But as you will see with “completing the square” later in this unit, this method is very general, because even if an expression does not look like a perfect square, you can usually **make** it look like one if you want to—and if you know how to spot the pattern.

The Difference Between Two Squares

The third type of factoring is based on the third of our basic formulae:

Equation:

$$(x + a)(x - a) = x^2 - a^2$$

You can run this formula in reverse whenever you are **subtracting two perfect squares**. For instance, if we see $x^2 - 25$, we recognize that both x^2 and 25 are perfect squares. We can therefore factor it as $(x + 5)(x - 5)$. Other examples include:

Example:

- $x^2 - 64 = (x + 8)(x - 8)$
- $16y^2 - 49 = (4y + 7)(4y - 7)$
- $2x^2 - 18 = 2(x^2 - 9) = 2(x + 3)(x - 3)$

And so on. Note that, in the last example, we begin by **pulling out** a 2, and we are then left with two perfect squares. This is an example of the rule that you should always **begin by pulling out common factors** before you try anything else!

It is also important to note that you cannot factor the sum of two squares. $x^2 + 4$ is a perfectly good function, but it cannot be factored.

Brute Force, Old-Fashioned, Bare-Knuckle, No-Holds-Barred Factoring

In this case, the multiplication that we are reversing is just FOIL. For instance, consider:

Equation:

$$(x + 3)(x + 7) = x^2 + 3x + 7x + 21 = x^2 + 10x + 21$$

What happened? The 3 and 7 **added** to yield the middle term (10), and **multiplied** to yield the final term (21). We can generalize this as:

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

The point is, if you are given a problem such as $x^2 + 10x + 21$ to factor, you look for two numbers that **add** up to 10, and **multiply** to 21. And how do you find them? There are a lot of pairs of numbers that add up to 10, but relatively few that multiply to 21. So you start by looking for factors of 21.

Below is a series of examples. Each example showcases a different aspect of the factoring process, so I would encourage you not to skip over any of them: try each problem yourself, then take a look at what I did.

If you are uncomfortable with factoring, the best practice you can get is to multiply things out. In each case, look at the final answer I arrive at, and multiply it with FOIL. See that you get the problem I started with. Then look back at the steps I took and see how they led me to that answer. The steps will make a lot more sense if you have done the multiplication already.

Exercise:

Problem: Factor $x^2 + 11x + 18$

$$(x + \underline{\hspace{1cm}})(x + \underline{\hspace{1cm}})$$

Solution:

What **multiplies** to 18? $1 \cdot 18$, or $2 \cdot 9$, or $3 \cdot 6$.

Which of those **adds** to 11? $2 + 9$.

$$(x + 2)(x + 9)$$

Note: Start by listing all factors of the third term. Then see which ones add to give you the middle term you want.

Exercise:

Problem: Factor $x^2 - 13x + 12$

$$(x + \underline{\hspace{1cm}})(x + \underline{\hspace{1cm}})$$

Solution:

What **multiplies** to 12? $1 \cdot 12$, or $2 \cdot 6$, or $3 \cdot 4$

Which of those **adds** to 13? $1 + 12$

$$(x - 1)(x - 12)$$

Note: If the middle term is negative, it doesn't change much: it just makes both numbers negative. If this had been $x^2 + 13x + 12$, the process would have been the same, and the answer would have been $(x + 1)(x + 12)$.

Exercise:

Problem: Factor $x^2 + 12x + 24$

$$(x + \underline{\hspace{1cm}})(x + \underline{\hspace{1cm}})$$

Solution:

What **multiplies** to 24? $1 \cdot 24$, or $2 \cdot 12$, or $3 \cdot 8$, or $4 \cdot 6$

Which of those **adds** to 12? None of them.

It can't be factored. It is "prime."

Note: Some things can't be factored. Many students spend a long time fighting with such problems, but it really doesn't have to take long. Try all the possibilities, and if none of them works, it can't be factored.

Exercise:

Problem: Factor $x^2 + 2x - 15$

$$(x + __)(x + __)$$

Solution:

What **multiplies** to 15? $1 \cdot 15$, or $3 \cdot 5$

Which of those **subtracts** to 2? $5-3$

$$(x + 5)(x - 3)$$

Note: If the **last** term is negative, that changes things! In order to **multiply** to -15 , the two numbers will have to have different signs—one negative, one positive—which means they will **subtract** to give the middle term. Note that if the **middle** term were negative, that wouldn't change the process: the final answer would be reversed, $(x + 5)(x - 3)$. This fits the rule that we saw earlier—changing the sign of the middle term changes the answer a bit, but not the process.

Exercise:

Problem: Factor $2x^2 + 24x + 72$

Solution:

$$2(x^2 + 12x + 36)$$

$$2(x + 6)^2$$

Note: Never forget, **always** start by looking for common factors to pull out. Then look to see if it fits one of our formulae. Only after trying all that do you begin the FOIL approach.

Exercise:

Problem: Factor $3x^2 + 14x + 16$

$$(3x + \underline{\hspace{1cm}})(x + \underline{\hspace{1cm}})$$

Solution:

What **multiplies** to 16? $1 \cdot 16$, or $2 \cdot 8$, or $4 \cdot 4$

Which of those adds to 14 **after tripling one number**? $8 + 3 \cdot 2$

$$(3x + 8)(x + 2)$$

Note: If the x^2 has a coefficient, and if you can't pull it out, the problem is trickier. In this case, we know that the factored form will look like $(3x + \underline{\hspace{1cm}})(x + \underline{\hspace{1cm}})$ so we can see that, when we multiply it back, one of those numbers—the one on the right—will be tripled, before they add up to the middle term! So you have to check the number pairs to see if any work that way.

Checking Your Answers

There are two different ways to check your answer after factoring: multiplying back, and trying numbers.

Example:

1. **Problem:** Factor $40x^3 - 250x$

- $10x(4x - 25)$ **First, pull out the common factor**
- $10x(2x + 5)(2x - 5)$ **Difference between two squares**

2. **So, does $40x^3 - 250x = 10x(2x + 5)(2x - 5)$? First let's check by multiplying back.**

- $10x(2x + 5)(2x - 5)$
- $= (20x^2 + 50x)(2x - 5)$ **Distributive property**
- $= 40x^3 - 100x^2 + 100x^2 - 250x$ **FOIL**
- $= 40x^3 - 250x$ ✓

3. **Check by trying a number. This should work for any number. I'll use $x = 7$ and a calculator.**

- $40x^3 - 250x \stackrel{?}{=} 10x(2x + 5)(2x - 5)$
- $40(7)^3 - 250(7) \stackrel{?}{=} 10(7)(2 \cdot 7 + 5)(2 \cdot 7 - 5)$
- $11970 = 11970$ ✓

I stress these methods of checking answers, not just because checking answers is a generally good idea, but because they reinforce key concepts. The first method reinforces the idea that **factoring is multiplication done backward**. The second method reinforces the idea of algebraic generalizations.

Quadratic Concepts -- Solving Quadratic Equations by Factoring

Consider the equation $4x^2 + 14x - 60 = 0$. This is **not** an algebraic generalization, but an equation to be solved for x : that is, it asks the question “What x value, or values, will make this equation true?” We will be solving such equations in three different ways. The fastest and easiest is by factoring.

Using the techniques discussed above, we can rewrite this problem as follows. (Try it for yourself!)

$$4x^2 + 14x - 60 = 0 \text{ **Original form**}$$

$$2(2x - 5)(x + 6 = 0) \text{ **Factored form**}$$

The second form may look more complicated than what we started with. But consider what this equation says. There are three numbers: 2, $2x - 5$, and $x + 6$. The equation says that when you multiply these three numbers, you get 0. Ask yourself this crucial question: **How can you multiply numbers and get the answer 0?**

The only way it can happen is if one of the numbers is 0. Take a moment to convince yourself of this: if several numbers **multiply** to give 0, one of those numbers must be 0.

So we have three possibilities.

$2 = 0$	$2x - 5 = 0$	$x + 6 = 0$
(it just isn't)	$x = 2\frac{1}{2}$	$x = -6$

The moral of the story is: when a quadratic equation is factored, it can be solved easily. In this case, the equation $4x^2 + 14x - 60 = 0$ has two valid solutions, $x = 2\frac{1}{2}$ and $x = -6$.

Consider this example:

Equation:

$$x^2 - 9x + 20 = 6$$

A common mistake is to solve it like this.

Example:

$x^2 - 9x + 20 = 6$, solved **incorrectly**

- $(x - 4)(x - 5) = 6$
- $(x - 4) = 6$

- $x = 10$ ✗

- $(x - 5) = 6$

- $x = 11$ ✗

All looks good, doesn't it? The factoring was correct. But if you try $x = 10$ or $x = 11$ in the original equation, you will find that neither one works. What went wrong?

The factoring was correct, but the next step was wrong. Just because $(x - 4)(x - 5) = 6$ does not mean that either $(x - 4)$ or $(x - 5)$ has to be 6. There are lots of ways for two numbers to multiply to give 6. This trick only works for 0!

Example:

$x^2 - 9x + 20 = 6$, solved **correctly**

- $x^2 - 9x + 14 = 0$
- $(x - 7)(x - 2) = 0$
- $(x - 7) = 0$

- $x = 7$

- $(x - 2) = 0$

- $x = 2$

You may want to confirm for yourself that these are the correct solutions.

Moral: When solving quadratic equations, always begin by moving **everything to one side of the equation**, leaving only a 0 on the other side. This is true regardless of which of the three methods you use.

Example:

$$x^2 + 14x + 49 = 0$$

- $(x + 7)^2 = 0$
- $x = -7$

Moral: If the left side factors as a perfect square, the quadratic equation has only one solution.

Not all quadratic functions can be factored. This does not mean they have no solutions! If the function cannot be factored, we must use other means to

find the solutions.

Quadratic Concepts -- Solving Quadratic Equations by Completing the Square

This module teaches the method of solving quadratic equations by completing the square.

Consider the equation:

Equation:

$$(x + 3)^2 = 16$$

We can solve this by analogy to the way that we approached absolute value problems. **something** squared is 16. So what could the **something** be? It could be 4. It could also be -4 . So the solution is:

- $x + 3 = 4$
 - $x = 1$
- $x + 3 = -4$
 - $x = -7$

These are the two solutions.

This simple problem leads to a completely general way of solving quadratic equations—because any quadratic equation can be put in a form like the above equation. The key is **completing the square** which, in turn, is based on our original two formulae:

Equation:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Equation:

$$(x - a)^2 = x^2 - 2ax + a^2$$

As an example, consider the equation $x^2 + 10x + 21 = 0$. In order to make it fit one of the patterns above, we must replace the 21 with the correct number: a number such that $x^2 + 10x + \underline{\hspace{1cm}}$ is a perfect square. What number goes there? If you are familiar with the pattern, you know the answer right away. 10 is 5 **doubled**, so the number there must be 5 **squared**, or 25.

But how do we turn a 21 into a 25? We add 4, of course. And if we add 4 to **one** side of the equation, we have to add 4 to the **other** side of the equation. So the entire problem is worked out as follows:

Example:

Solving by Completing the Square (quick-and-dirty version)

Solve $x^2 + 10x + 21 = 0$

The problem.

$x^2 + 10x + 25 = 4$

Add 4 to both sides, so that the left side becomes a perfect square.

$(x + 5)^2 = 4$

Rewrite the perfect square.

$x + 5 = 2$
 $x = -3$

$x + 5 = -2$
 $x = -7$

If something-squared is 4, the something can be 2, or -2 . Solve both possibilities to find the two answers.

Thus, we have our two solutions.

Completing the square is more time-consuming than factoring: so whenever a quadratic equation can be factored, factoring is the preferred method. (In this case, we would have factored the original equation as $(x + 3)(x + 7)$ and gotten straight to the answer.) However, completing the square can be used on **any** quadratic equation. In the example below, completing the square is used to find two answers that would not have been found by factoring.

Example:

Solving by Completing the Square (showing all the steps more carefully)

Solve $9x^2 - 54x + 80 = 0$

The problem.

$9x^2 - 54x = -80$

Put all the x terms on one side, and the number on the other

$x^2 - 6x = -\frac{80}{9}$

Divide both sides by the coefficient of x^2 (*see below)

$x^2 - 6x + 9 = -\frac{80}{9} + \underline{9}$

Add the same number (*see below) to both sides, so that the left side becomes a perfect square.

$(x - 3)^2 = -\frac{80}{9} + \frac{81}{9} = \frac{1}{9}$

Rewrite the perfect square.

Solving by Completing the Square (showing all the steps more carefully)

$$x - 3 = \frac{1}{3}$$

$$x = 3\frac{1}{3}$$

$$x - 3 = -\frac{1}{3}$$

$$x = 2\frac{2}{3}$$

If something-squared is $\frac{1}{9}$, the something can be $\frac{1}{3}$, or $-\frac{1}{3}$. Solve both possibilities to find the two answers.

Two steps in particular should be pointed out here.

In the third step, we divide both sides by 9. When completing the square, you do not want to have any coefficient in front of the term; if there is a number there, you divide it out. Fractions, on the other hand (such as the $-\frac{80}{9}$ in this case) do not present a problem. This is in marked contrast to factoring, where a coefficient in front of the x^2 can be left alone, but fractions make things nearly impossible.

The step after that is where we actually complete the square. $x^2 + 6x + \underline{\hspace{1cm}}$ will be our perfect square. How do we find what number we want? Start with the coefficient of x^2 (in this case, 6). **Take half of it, and square the result.** Half of 6 is 3, squared is 9. So we want a 9 there to create $x^2 + 6x + 9$ which can be simplified to $(x + 3)^2$.

$$x^2 - 6x + 9$$

Take half of this and put it here $\rightarrow$ $(x - 3)^2$ $\rightarrow$ Then square that and put it here

If the coefficient of x is an **odd number**, the problem becomes a little uglier, but the principle is the same. For instance, faced with:

Equation:

$$x^2 + 5x + \underline{\hspace{1cm}}$$

You would begin by taking half of 5 (which is $\frac{5}{2}$) and then squaring it:

$$x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$$

Another “completing the square” example, in which you cannot get rid of the square root at all, is presented in the worksheet “The Generic Quadratic Equation.”

One final note on completing the square: there are three different possible outcomes.

- If you end up with something like $(x - 3)^2 = 16$ you will find two solutions, since $x - 3$ can be either 4, or -4 . You will always have two solutions if the right side of the equation, after completing the square, is positive.
- If you end up with $(x - 3)^2 = 0$ then there is only one solution: x must be 3 in this example. If the right side of the equation is 0 after completing the square, there is only one solution.
- Finally, what about a negative number on the right, such as $(x - 3)^2 = -16$? Nothing squared can give a negative answer, so there is no solution.

Quadratic Concepts -- The Quadratic Formula

This module discusses the quadratic formula.

In "Solving Quadratic Equations by Completing the Square" I talked about the common mathematical trick of solving a problem once, **using letters instead of numbers**, and then solving specific problems by plugging numbers into a general solution.

In the text, you go through this process for quadratic equations in general. The definition of a quadratic equation is any equation that can be written in the form:

Equation:

$$ax^2 + bx + c = 0$$

where $a \neq 0$. By completing the square on this generic equation, you arrive at the quadratic formula:

Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula can then be used to solve any quadratic equation, without having to complete the square each time. To see how this formula works, let us return to the previous problem:

Equation:

$$9x^2 - 54x + 80 = 0$$

In this case, $a = 9$, $b = -54$, and $c = 80$. So the quadratic formula tells us that the answers are:

Equation:

$$x = \frac{-(-54) \pm \sqrt{(-54)^2 - 4(9)(80)}}{2(9)}$$

We'll use a calculator here rather than squaring 54 by hand....

Equation:

$$x = \frac{54 \pm \sqrt{2916 - 2880}}{18} = \frac{54 \pm \sqrt{36}}{18} = \frac{54 \pm 6}{18} = \frac{9 \pm 1}{3}$$

So we find that the two answers are $\frac{10}{3}$ and $\frac{8}{3}$, which are the same answers we got by completing the square.

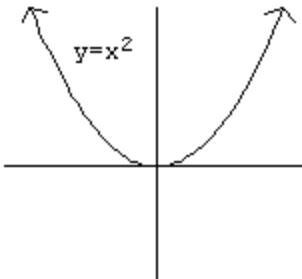
Using the quadratic formula is usually faster than completing the square, though still slower than factoring. So, in general, try to factor first: if you cannot factor, use the quadratic formula.

So why do we learn completing the square? Two reasons. First, completing the square is how you **derive** the quadratic formula. Second, completing the square is vital to **graphing** quadratic functions, as you will see a little further on in the chapter.

Quadratic Concepts -- Graphing Quadratic Equations

This module covers the graphing of quadratic equations.

The graph of the simplest quadratic function, $y = x^2$, looks like this:

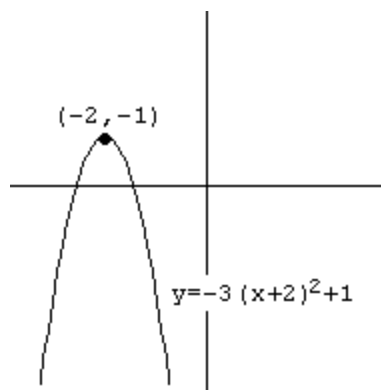


(You can confirm this by plotting points.) The point at the bottom of the U-shaped curve is known as the “vertex.”

Now consider the function $y = -3(x + 2)^2 + 1$. It’s an intimidating function, but we have all the tools we need to graph it, based on the **permutations** we learned in the first unit. Let’s step through them one by one.

- What does the $-$ sign do? It multiplies all y -values by -1 ; positive values become negative, and vice-versa. So we are going to get an **upside-down** U-shape. We say that $y = x^2$ “opens up” and $y = -x^2$ “opens down.”
- What does the 3 do? It multiplies all y -values by 3; positive values become more positive, and negative values become more negative. So it **vertical stretches** the function.
- What does the $+1$ at the end do? It adds 1 to all y -values, so it moves the function **up** by 1.
- Finally, what does the $+2$ do? This is a horizontal modification: if we plug in $x = 10$, we will be evaluating the function at $x = 12$. In general, we will always be copying the original x^2 function to our **right**; so we will be 2 units to the **left** of it.

So what does the graph look like? It has moved 2 to the left and 1 up, so the vertex moves from the origin (0,0) to the point $(-2,1)$. The graph has also flipped upside-down, and stretched out vertically.



So graphing quadratic functions is easy, no matter how complex they are, if you understand permutations—and if the functions are written in the form $y = a(x - h)^2 + k$, as that one was.

Graphing Quadratic Functions

The graph of a quadratic function is always a vertical parabola. If the function is written in the form $y = a(x - h)^2 + k$ then the vertex is at (h, k) . If a is positive, the parabola opens up; if a is negative, the parabola opens down.

But what if the functions are not expressed in that form? We're more used to seeing them written as $y = ax^2 + bx + c$. For such a function, you graph it by first putting it into the form we used above, and then graphing it. And the way you get it into the right form is...completing the square! This process is almost identical to the way we used completing the square to **solve** quadratic equations, but some of the details are different.

Example:

Graphing a Quadratic Function

Graphing a Quadratic Function

Graph $2x^2 - 20x + 58$

The problem.

$$2(x^2 - 10x) + 58$$

We used to start out by dividing both sides by the coefficient of x^2 (2 in this case). In this case, we don't have another side: we can't make that 2 go away. But it's still in the way of completing the square. So we factor it out of the first two terms. Do not factor it out of the third (numerical) term; leave that part alone, outside of the parentheses.

$$2(x^2 - 10x + 25) + 58 - 50$$

Inside the parentheses, add the number you need to complete the square. (Half of 10, squared.) Now, when we add 25 inside the parentheses, what we have really done to our function? We have added 50, since everything in parentheses is doubled. So we keep the function the same by subtracting that 50 right back again, outside the parentheses! Since all we have done in this step is add 50 and then subtract it, the function is unchanged.

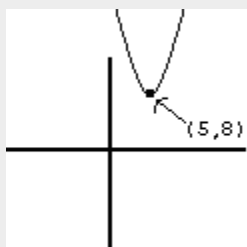
Graphing a Quadratic Function

$$2(x - 5)^2 + 8$$

Inside the parentheses, you now have a perfect square and can rewrite it as such. Outside the parentheses, you just have two numbers to combine.

Vertex (5,8) opens up

Since the function is now in the correct form, we can read this information straight from the formula and graph it. Note that the number inside the parentheses (the h) always changes sign; the number outside (the k) does not.



So there's the graph! It's easy to draw once you have the vertex and direction. It's also worth knowing that the 2 vertically stretches the graph, so it will be thinner than a normal x^2 .

This process may look intimidating at first. For the moment, don't worry about mastering the whole thing—instead, look over every **individual step** carefully and make sure you understand why it works—that is, why it keeps the function fundamentally unchanged, while moving us toward our goal of a form that we can graph.

The good news is, this process is basically the same every time. A different example is worked through in the worksheet “Graphing Quadratic

Functions II”—that example differs only because the x^2 term does not have a coefficient, which changes a few of the steps in a minor way. You will have plenty of opportunity to practice this process, which will help you get the “big picture” if you understand all the individual steps.

And don’t forget that what we’re really creating here is an algebraic generalization!

Equation:

$$2x^2 - 20x + 58 = 2(x - 5)^2 + 8$$

This is exactly the sort of generalization we discussed in the first unit—the assertion that these two very different functions will **always give the same answer** for any x -value you plug into them. For this very reason, we can also assert that the two graphs will look the same. So we can graph the first function by graphing the second.

Quadratic Concepts -- Solving Problems by Graphing Quadratic Functions

Surprisingly, there is a fairly substantial class of real world problems that can be solved by graphing quadratic functions.

These problems are commonly known as “optimization problems” because they involve the question: “When does this important function reach its **maximum**?” (Or sometimes, its **minimum**?) In real life, of course, there are many things we want to maximize—a company wants to maximize its revenue, a baseball player his batting average, a car designer the leg room in front of the driver. And there are many things we want to minimize—a company wants to minimize its costs, a baseball player his errors, a car designer the amount of gas used. Mathematically, this is done by writing a function for that quantity and finding where that function reaches its highest or lowest point.

Example:

Exercise:

Problem:

If a company manufactures x items, its total cost to produce these items is $x^3 - 10x^2 + 43x$. How many items should the company make in order to **minimize** its **average cost per item**?

Solution:

“Average cost per item” is the total cost, divided by the number of items. For instance, if it costs \$600 to manufacture 50 items, then the average cost per item was \$12. It is important for companies to minimize average cost because this enables them to sell at a low price.

In this case, the total cost is $x^3 - 10x^2 + 43x$ and the number of items is x . So the **average cost per item** is

Equation:

$$A(x) = \frac{x^3 - 10x^2 + 43x}{x} = x^2 - 10x + 43$$

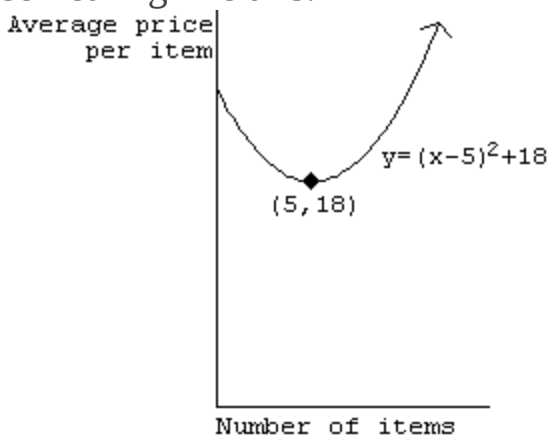
What the question is asking, mathematically, is: what value of x makes this function the **lowest**?

Well, suppose we were to graph this function. We would complete the square by rewriting it as:

Equation:

$$A(x) = x^2 - 10x + 25 + 18 = (x - 5)^2 + 18$$

The graph opens up (since the $(x - 5)^2$ term is positive), and has its vertex at $(5, 18)$ (since it is moved 5 to the right and 18 up). So it would look something like this:



I have graphed only the first quadrant, because negative values are not relevant for this problem (why?).

The real question here is, what can we learn from that graph? Every point on that graph represents one possibility for our company: if they manufacture x items, the graph shows what $A(x)$, the average cost per item, will be.

The point $(5,18)$ is the **lowest point on the graph**. It is possible for x to be higher or lower than 5, but it is never possible for A to be lower than 18. So if its goal is to minimize average cost, their best strategy is to manufacture 5 items, which will bring their average cost to \$18/item.